Mathematical Formulations

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This documents defines mathematical formulations referenced in the paper:

> Mahmoud Parham, Thomas Fenz, Nikolaus Süss, Klaus-Tycho Foerster, and Stefan Schmid. Traffic engineering with joint link weight and segment optimization. In *CoNext'21*, 2021

1 Mixed-Integer Linear Program for Joint

Minimizing Maximum Link Utilization

Minimize MLU

Constants and Variables

$$\begin{cases} M := \max\{\sum_{(s,t,d)\in\mathcal{D}} d, 2|E|\} & \triangleright \text{ “Large M”} \\ \mathcal{S} := \{(p,q) \in V^2 \mid p \neq q\} & \triangleright \text{ Segments} \end{cases} \tag{1}$$

$$\begin{cases} S_\delta^{(p,q)}, x_\ell^t \in \{0,1\}, \\ f_\ell^{(p,q)}, D^{(p,q)}, dist_p^q \geq 0, \\ 0 < \omega_\ell \leq 1 \end{cases} \qquad \begin{cases} (p,q) \in \mathcal{S}, \ell \in E, \\ \delta = (s,t,d) \in \mathcal{D} \end{cases} \tag{2}$$

Flow Conservation

$$\sum_{\ell=(*,v)\in E} f_\ell^{(p,q)} - \sum_{\ell=(v,*)\in E} f_\ell^{(p,q)} = \begin{cases} 0 & v \neq p,q \\ -D^{(p,q)} & v = p \\ D^{(p,q)} & v = q \end{cases} \qquad \begin{cases} (p,q) \in \mathcal{S}, \\ v \in V \end{cases} \tag{3}$$

$$\sum_{(p,q)\in\mathcal{S}} f_\ell^{(p,q)} \leq MLU \cdot C_\ell \qquad \ell \in ELP : capacity \tag{4}$$

Waypoint Setting

$$D^{(p,q)} = \sum_{\delta=(s,t,d)\in\mathcal{D}} S_\delta^{(p,q)} \cdot d \qquad (p,q) \in \mathcal{S} \tag{5}$$

$$\sum_{(p,q)\in\mathcal{S}, q=v} S_D^{(p,q)} - \sum_{(p,q)\in\mathcal{S}, p=v} S_D^{(p,q)} = \begin{cases} 0 & v \neq s,t \\ -1 & v = s \\ 1 & v = t \end{cases} \qquad \begin{cases} \delta = (s,t,d) \in \mathcal{D}, \\ v \in V \end{cases} \tag{6}$$

$$\sum_{(p,q)\in\mathcal{S}} S_D^{(p,q)} \leq W + 1 \qquad \delta = (s,t,d) \in \mathcal{D} \tag{7}$$

Even Splitting

$$f_\ell^{(p,q)} \leq M \cdot x_\ell^q \qquad (p,q) \in \mathcal{S}, \ell \in E \tag{8}$$

$$\sum_{\ell=(v,*)\in E} x_\ell^t \leq \text{SF} \qquad v,t \in V, v \neq t \tag{9}$$

$$\begin{cases} \sum_{p:(p,t)\in\mathcal{S}} f_\ell^{(p,t)} \leq f_v^t \\ f_v^t - \sum_{p:(p,t)\in\mathcal{S}} f_\ell^{(p,t)} \leq M \cdot (1 - x_\ell^t) \end{cases} \qquad \begin{cases} v,t \in V, v \neq t, \\ \ell = (v,*) \end{cases} \tag{10}$$

Weight Setting

$$\begin{cases} dist_u^t \leq dist_v^t + \omega_\ell \\ dist_v^t - dist_u^t + \omega_\ell \leq M \cdot (1 - x_\ell^t) \\ 1 - x_\ell^t \leq M \cdot (dist_v^t - dist_u^t + \omega_\ell) \end{cases} \qquad \begin{cases} \ell = (u,v), \\ t \in V, t \neq u \end{cases} \tag{11}$$

Variables

Line (1) defines two LP constants. The constant M is a large number used in place of ∞. The set $\mathcal{S}$ consists of all node pairs, where each pair represents a possible segment. The variables defined at (2) are described as follows.

SF: the splitting factor. A flow splits into at most SF equal size parts.

W: maximum number of waypoints allowed per demand.

M: a constant large enough.

x_ℓ^t: binary variables indicating whether link ℓ is on a shortest path to node t.

$f_\ell^{(p,q)}$: the fractional amount of flow of the segment (p, q) assigned to the link ℓ.

$f_v^{(p,q)}$: the total flow of segment (p, q) leaving node v.

$S_\delta^{(p,q)}$: binary variable indicating whether the segment (p, q) is active for the demand $\delta = (s, t, d) \in \mathcal{D}$.

$dist_v^t$: shortest path weight from v to t.

ω_ℓ: the weight of the link ℓ.

Constraints

- (3)-(4) enforce flow conservation and capacity rules for the flow each segment separately. The flow size depends on sizes of demands for which the segment is active, hence a variable.
- (5) for each segment $(p, q) \in \mathcal{S}$, determines the total flow size that must traverse between the endpoints of the segment. The size of a demand is added to this quantity only if the segment is active for that demand.

  (6) ensures segments active for any demand are consecutive and connect the endpoints of that demand. (7) limits the number of active segments for a demand, i.e., the number of waypoints. If $W = 0$ then the MILP is the formulation of WPO.
- (8) ensures a link will be on a shortest path to q if it is assigned a positive flow for any segment ending at q.
- (9) limits the splitting factor, that is, the number of parts a flow is allowed to split into. E.g., if SF = 1 then no flow splits in any feasible solution. (10) ensures a flow splits evenly on all adjacent links that are set to be on a shortest path to t.
- (11) determines the weight of each link that is set be on a shortest path, by determining a weighted distance ("node potentials") for its endpoints.

2 Maximal Multi-commodity Flow (MCF)

Assume a set of source-terminal pairs is given in $\mathcal{D}$. The following LP computes a demand size for each pair such that the sum of all demands is maximized.

$$\text{Maximize} \sum_{(s,t)\in\mathcal{D}} d^{(s,t)} \tag{12}$$

$$f_\ell^{(s,t)}, d^{(s,t)} \geq 0 \qquad (s,t)\in\mathcal{D}, \ell \in E \tag{13}$$

$$\sum_{\ell=(*,v)\in E} f_\ell^{(s,t)} - \sum_{\ell=(v,*)\in E} f_\ell^{(s,t)} = \begin{cases} 0 & v \neq s,t \\ -d^{(s,t)} & v = s \\ d^{(s,t)} & v = t \end{cases} \qquad \begin{cases} (s,t)\in\mathcal{D}, \\ v \in V \end{cases} \tag{14}$$

$$\sum_{(s,t)\in\mathcal{D}} f_\ell^{(s,t)} \leq C_\ell \qquad \ell \in E \tag{15}$$

3 Maximum Concurrent Multi-commodity Flow

Assume demands are given and $d^{(s,t)}$ denotes the size of the demand from s to t for each source-terminal pair (s,t). The next LP computes a scale factor F that when we multiply to every demand size, the sum of new demand sizes equals the size of the maximum concurrent MCF.

$$\text{Maximize } F \tag{16}$$

$$f_\ell^{(s,t)}, F \geq 0 \qquad (s,t)\in\mathcal{D}, \ell \in E \tag{17}$$

$$\sum_{\ell=(*,v)\in E} f_\ell^{(s,t)} - \sum_{\ell=(v,*)\in E} f_\ell^{(s,t)} = \begin{cases} 0 & v \neq s,t \\ -d^{(s,t)} \cdot F & v = s \\ d^{(s,t)} \cdot F & v = t \end{cases} \qquad \begin{cases} (s,t)\in\mathcal{D}, \\ v \in V \end{cases} \tag{18}$$

$$\sum_{(s,t)\in\mathcal{D}} f_\ell^{(s,t)} \leq C_\ell \qquad \ell \in E \tag{19}$$